Chapter 1

Set theory

1.1 Sets

Definition 1.1. So it begins...

- A **set** is a collection of unique objects;
- The objects in a set are called its **elements**.

Sets are usually denoted by uppercase Roman letters, and one way to write down a set is to list its elements inside curly braces:

$$A = \{1,\ 2,\ 3\}$$
$$B = \{a,\ b,\ c\}$$
$$C = \left\{a,\ \pi,\ f(x) = x^2,\ \right.$$  $$\left.\right\}$$
$$H = \{\text{Hello Dolly!},\ \text{La Cage aux Folles}\}$$

Literally anything could get thrown into a set. Even another set, or set of sets, or set of set of sets, etc. The only thing we would *not* write is something like $\{1,\ 2,\ 2,\ 3\}$, because the objects in a set must be unique. So no repeats.

Notational Point 1.1. $2 \in A$ is shorthand for "2 is an element of the set A," and $4 \notin A$ is shorthand for "4 is not an element of the set A." The symbol "$\notin$" is like "$\neq$."

Definition 1.2. A set with one element in it, like $D = \{4\}$, is called a **singleton set**.

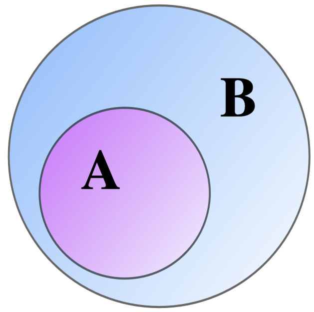

Figure 1.1: Some set B, with a subset A squatting inside.

1.2 Famous sets

Here are some examples of special sets that we give names to:

$$\begin{aligned}
\mathbb{N} &= \{0, 1, 2, ...\} && \text{(natural numbers)}\\
\mathbb{Z} &= \{..., -2, -1, 0, 1, 2, ...\} && \text{(integers)}\\
\mathbb{Q} &= \{\text{“ratios of integers”}\} && \text{(rational numbers)}\\
\mathbb{R} &= \{\text{“}\mathbb{Q}\text{ plus irrationals like }\pi, e, \sqrt{2}\text{, and friends"}\} && \text{(real numbers)}\\
\mathbb{C} &= \{a + bi \,:\, a, b \in \mathbb{R}, i^2 = -1\} && \text{(complex numbers)}\\
\varnothing &= \{\} && \text{(empty set)}
\end{aligned}$$

The starting point in mathematical probability is to write down the set S of possible outcomes of a random phenomenon:

$$\begin{aligned}
&\{H, T\} && \text{(Coin flip)}\\
&\{1, 2, 3, 4, 5, 6\}. && \text{(Die roll)}
\end{aligned}$$

This is called the **sample space**. If we wish to study phenomenon where we are uncertain what will happen, the first thing we must do is specify the realm of possibilities. What *could* happen? We write this down with a set.

Remark 1.1. We see from the examples thus far that sets can be either **finite** or **infinite**, and among infinite sets there are the **countable** sets like $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$, and the **uncountable** ones like $\mathbb{R}$ and $\mathbb{C}$.

1.3 Subsets

Definition 1.3. A is a **subset** of B if all of the elements in A are in B. We denote this $A \subseteq B$, read “A is a subset of B." See Figure 1.1 for a cartoon.

Consider

$$\begin{aligned}
A &= \{1, 2\}\\
B &= \{1, 2, 3\}\\
C &= \{1, 2, 3\}.
\end{aligned}$$

We see that $A \subseteq B$, $A \subseteq C$, $B \subseteq C$, and $C \subseteq B$. Indeed, the last pair of relations suggest a definition:

Definition 1.4. $B = C$ if $B \subseteq C$ and $C \subseteq B$.

We see then that the notation "$\subseteq$" is not *strict*, and is meant in the same spirit as "$\leq$."

Recalling our famous sets, we have

- $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$;
- $\varnothing \subseteq A$ for any set A.

In a probability context, the sample space S is the set of all possible outcomes of a random phenomenon, and subsets of the sample space are called **events**. These are the things we will try to compute the probabilities of. Recalling the die roll example, the sample space is $S = \{1, 2, 3, 4, 5, 6\}$, and the event that "you roll an odd number" can be represented by the subset $\{1, 3, 5\}$.

1.4 Subsets of the real line

The familiar interval notation is just shorthand for subsets of the real line:

$$\begin{aligned}
(a,\ b) &= \{x \in \mathbb{R} \,:\, a < x < b\} \\
[a,\ b] &= \{x \in \mathbb{R} \,:\, a \leq x \leq b\} \\
(a,\ b] &= \{x \in \mathbb{R} \,:\, a < x \leq b\} \\
[a,\ b) &= \{x \in \mathbb{R} \,:\, a \leq x < b\}.
\end{aligned}$$

These examples provide a model for an alternative method of writing down sets, especially for infinite sets where you cannot just list out all the elements. Consider writing down the set of odd numbers:

$$\begin{aligned}
O &= \{1,\ 3,\ 5,\ ...\} && \text{(Faking it)} \\
&= \{n \in \mathbb{N} \,:\, \exists k \in \mathbb{N} \text{ s.t. } n = 2k + 1\} && \text{(Completely rigorous)} \\
&= \{n \in \mathbb{N} \,:\, n \text{ is odd}\} && \text{(Good enough; less pompous)}
\end{aligned}$$

The basic template is $\{x \in S \,:\, x \text{ satisfies some condition}\}$, where S is the superset that your new subset is contained in.

1.5 Set operations

The three main operations on sets are:

- (**Union**) $A \cup B$ (read "A union B") is the set of elements in either A or B (or both!):

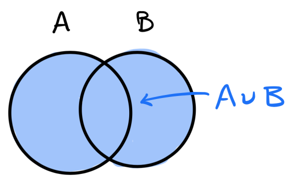

Figure 1.2: Union is like OR.

If $A = \{1, 2, 3\}$ and $B = \{1, 4, 5\}$, then $A \cup B = \{1, 2, 3, 4, 5\}$. Notice that 1 was not duplicated, because sets are collections of *unique* objects. When you plan your wedding, there are two lists: people you *want* to invite, and people you *have* to invite. When you merge them into the master list of invitees, grandma's name doesn't appear twice just because she was on both. Got it?

- (**Intersection**) $A \cap B$ (read "A intersect B") is the set of elements in both A and B:

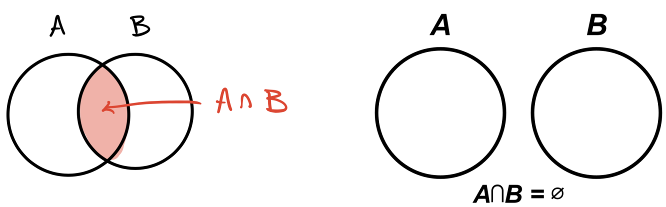

Figure 1.3: Intersection is like AND.

In the previous example, $A \cap B = \{1\}$. It could be the case that sets A and B do not have any elements in common, like on the right of Figure 1.3. Then $A \cap B = \varnothing$, and we call A and B **disjoint**. If H is the set of all Jerry Herman musicals JZ has bought tickets for, and M is the set of all musicals JZ has seen live, then sadly $H \cap M = \varnothing$. Don't ask;

- (**Complement**) If S denotes some overarching "reference set," and $A \subseteq S$, then $A^c = \{x \in S : x \notin A\}$. We read this "$A$ complement." It's the stuff *not in* A:

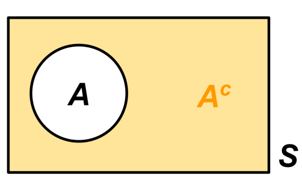

Figure 1.4: Complement is like NOT.

If $S = \mathbb{R}$ and $A = (-\infty, 2]$, then $A^c = (2, \infty)$.

Here are some simple but very useful facts about the set operations. Make sure you believe them!

Theorem 1.1. Let $A \subseteq B \subseteq S$ for some reference set S. Then:

- $A \cup B = B$;
- $A \cap B = A$;
- $(A^c)^c = A$;
- $A \cup A^c = S$;
- $A \cap A^c = \varnothing$;
- $A \cap \varnothing = \varnothing$;
- $A \cup \varnothing = A$.

Those last two are really just a special case of the first two. Remember why?

1.6 Algebraic properties

The basic set operations have the following algebraic properties:

Commutative	$A \cup B = B \cup A$ $A \cap B = B \cap A$
Associative	$(A \cup B) \cup C = A \cup (B \cup C)$ $(A \cap B) \cap C = A \cap (B \cap C)$
Distributive	$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
De Morgan's Laws	$(A \cup B)^c = A^c \cap B^c$ $(A \cap B)^c = A^c \cup B^c$.

Pay *very* close attention to De Morgan's laws. They say that the complement **does not** distribute over unions and intersections. Instead, it has its own special behavior that you simply must get used to.

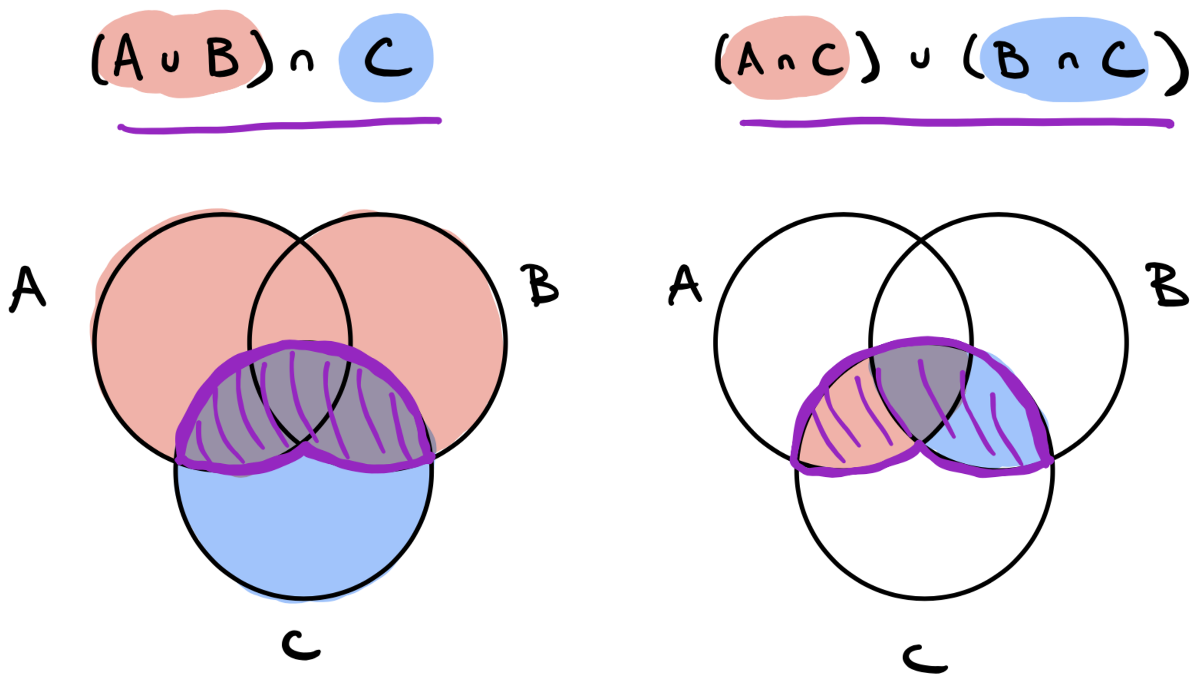

Figure 1.5

1.7 Proof techniques

The set identities above are actually theorems that we can prove. Here are some techniques:

1.7.1 Non-rigorous proof: Venn diagrams

Theorem 1.2. $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.

Demonstration. See Figure 1.5. We draw a picture representing the left-hand side of the equality, we draw a picture representing the right-hand side, and we recognize that in either case we drew the same thing. □

This is not rigorous because it's really just an example. We are representing sets with blobs in the plane and showing that things work in this case. Not all sets are blobs in the plane, and so strictly speaking this is not exhaustive. Nevertheless, it can be perfectly convincing for all practical purposes, and it often provides an excellent blueprint for how the rigorous proof will go.

1.7.2 Rigorous proof: element chasing

In an element-chasing argument, you select some arbitrary, representative element $x \in E$ and show that it must be the case that $x \in F$. So you use logic to "chase" the element x from the set E into the set F. Because x was arbitrary to begin with, the argument must hold for all $x \in E$, which establishes that $E \subseteq F$. If you can do this in both directions, you establish that $E = F$.

Theorem 1.3. $(A \cup B)^c = A^c \cap B^c$.

Proof. We prove subsethood in both directions:

- Show $(A \cup B)^c \subseteq A^c \cap B^c$. Let $x \in (A \cup B)^c$ be an arbitrary element. Then

$$\begin{aligned} x \in (A \cup B)^c &\implies x \notin A \cup B \\ &\implies x \notin A \text{ and } x \notin B \\ &\implies x \in A^c \text{ and } x \in B^c \\ &\implies x \in A^c \cap B^c. \end{aligned}$$

  Since x is an arbitrary element, we have the result for all elements in $(A \cup B)^c$, which necessarily implies subsethood.

- Show $A^c \cap B^c \subseteq (A \cup B)^c$. Let $x \in A^c \cap B^c$ be an arbitrary element. Then

$$\begin{aligned} x \in A^c \cap B^c &\implies x \in A^c \text{ and } x \in B^c \\ &\implies x \notin A \text{ and } x \notin B \\ &\implies x \notin A \cup B \\ &\implies x \in (A \cup B)^c. \end{aligned}$$

  Since x is an arbitrary element, we have the result.

Since $(A \cup B)^c \subseteq A^c \cap B^c$ and $A^c \cap B^c \subseteq (A \cup B)^c$, they are equal. □

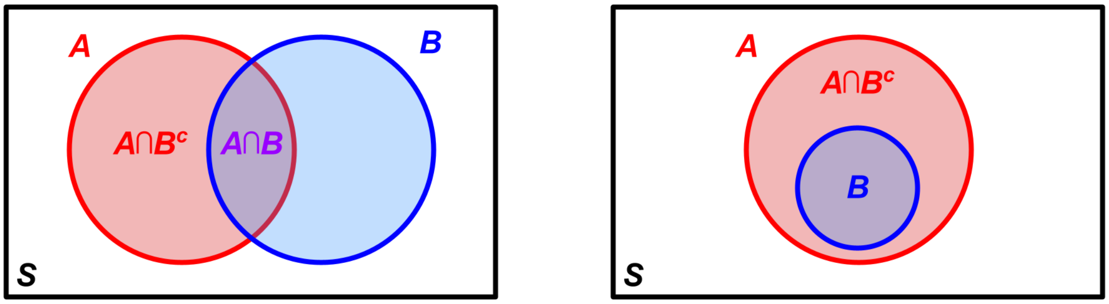

Figure 1.6

1.7.3 Rigorous proof: plain ol' algebra

Going forward, we can take for granted identities like the ones in Section 1.6. Armed with them, it is possible to prove subsequent set identities by simply stringing together algebraic operations in the same way that we would manipulate equations of real numbers. Take for example Figure 1.6, which motivates some simple but useful results.

Theorem 1.4. If $A,\ B \subseteq S$, then $A = (A \cap B) \cup (A \cap B^c)$, and this union is disjoint.

Proof. You can start on either side of the alleged equality and manipulate the expression until it turns into the other side. One direction:

$$\begin{aligned} A &= A \cap S && (A \subseteq S) \\ &= A \cap (B \cup B^c) && (B \cup B^c = S) \\ &= (A \cap B) \cup (A \cap B^c). && \text{(distributive property)} \end{aligned}$$

The other direction:

$$\begin{aligned} (A \cap B) \cup (A \cap B^c) &= A \cap (B \cup B^c) && \text{(distributive property in reverse)} \\ &= A \cap S && (B \cup B^c = S) \\ &= A. && (A \subseteq S) \end{aligned}$$

Lastly, we can do some algebra to see that the sets in this union are disjoint:

$$\begin{aligned} (A \cap B) \cap (A \cap B^c) &= (A \cap B) \cap (B^c \cap A) && \text{(commute)} \\ &= A \cap (B \cap B^c) \cap A && \text{(associate)} \\ &= A \cap (B \cap B^c) \\ &= A \cap \varnothing \\ &= \varnothing. \end{aligned}$$

□

Corollary 1.5. If $B \subseteq A \subseteq S$, then $A = B \cup (A \cap B^c)$, and this union is disjoint.

Proof. If $B \subseteq A$, then $A \cap B = B$, so it follows from Theorem 1.4. □