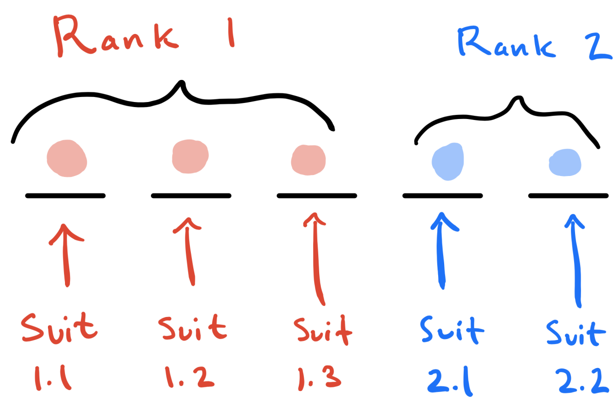

Figure 2.6

2.4.3 Examples

Example 2.8. (**Probability of a full house**) If a five-card hand is randomly dealt to you from a standard deck of 52 playing cards, what is the probability that you are dealt a *full house*: 3 cards of one rank and 2 cards of another rank? In words, our sample space and our event are S = "the set of all possible five-card hands" and A = "your hand is a full house." We have a deck with only 52 cards, and assuming it is adequately shuffled, no card is favored over any other, so we have a finite sample space with equally-likely outcomes. Therefore, we will ultimately apply $P(A) = \#(A)/\#(S)$. To compute $\#(S)$, we have to count all of the ways that five cards could be selected from a deck of 52. So we are "selecting $k = 5$ from $n = 52$." But does order matter, and are we drawing with replacement?

- We are sampling **without replacement**. Once a card is dealt to you, it stays in your hand, it does not go back in the deck to potentially be drawn again;
- **Order does not matter**. The only thing distinguishing one hand from another hand is the raw contents. The order in which the cards were dealt to you is irrelevant.

So, we are selecting $k = 5$ from $n = 52$ without replacement when order does not matter. This means that $\#(S) = \binom{52}{5}$. Great! Now what about $\#(A)$? To specify a full house, we need to determine four features:

- Rank 1 for three of the cards;
- Rank 2 for two of the cards;
- The suits of the Rank 1 cards;
- The suits of the Rank 2 cards.

This is pictured in Figure 2.6. We can think of each of these features as an “experiment" in the sense of the counting principle. So once we count the total number of ways Rank 1 can be selected, and Rank 2 can be selected, and the Rank 1 suits can be selecting, and the Rank 2 suits can be selected, we just have to multiply them together. So:

- There are 13 ways to choose Rank 1;
- Rank 2 must be distinct from Rank 1, so once Rank 1 is determined, there are 12 ways to choose Rank 2;
- Among the Rank 1 cards, we are drawing without replacement and order is irrelevant, so there are $\binom{4}{3} = 4$ ways to choose the three suits;
- Among the Rank 2 cards, we are drawing without replacement and order is irrelevant, so there are $\binom{4}{2} = 6$ ways to choose the two suits.

Given this, the counting principle tells us that the total number of ways to specify a full house is

$$\#(A) = 13 \cdot 12 \cdot 4 \cdot 6 = 3744$$

So the probability of interest is:

$$P(A) = \frac{\#(A)}{\#(S)} = \frac{3744}{\binom{52}{5}} \approx 0.00144.$$

Example 2.9. (**Probability that a pair has the same suit**) If a two-card hand is randomly dealt to you from a standard deck of 52 playing cards, what is the probability that you are dealt two cards of the same suit? In words, our sample space and our event are S = "the set of all possible two-card hands" and A = "your two cards are the same suit." Similar to the last example, we are sampling without replacement and order doesn't matter, so $\#(S) = \binom{52}{5} = 1326$. The way to proceed to is realize that we can rewrite the event A as a disjoint union. The event that "your two cards are the same suit" is the same as saying "you get a pair of hearts OR you get a pair of diamonds OR you get a pair of spades OR you get a pair of clubs." A hand could not be both a pair of clubs and a pair of spades, so this is a disjoint union

$$A = \{\text{pair } \heartsuit\} \cup \{\text{pair } \diamondsuit\} \cup \{\text{pair } \spadesuit\} \cup \{\text{pair } \clubsuit\},$$

and the axiom of countable additivity tells us that

$$P(A) = P(\text{pair } \heartsuit) + P(\text{pair } \diamondsuit) + P(\text{pair } \spadesuit) + P(\text{pair } \clubsuit).$$

Furthermore, it pays to recognize that there are the same number of cards of each suit in the deck. Because outcomes are equally likely, no suit is favored, so whatever they are, the summands in the above expression are all equal. So we only have to compute one of them:

$$P(A) = 4 \cdot P(\text{pair } \heartsuit).$$

So, what is the probability of being dealt a pair of hearts? Well, there are 13 hearts in the deck, and we are selecting two of them without replacement and ignoring order. So $\#(\text{pair } \heartsuit) = \binom{13}{2} = 78$, and the probability of interest is

$$P(A) = 4 \cdot \frac{78}{1326} \approx 0.059.$$

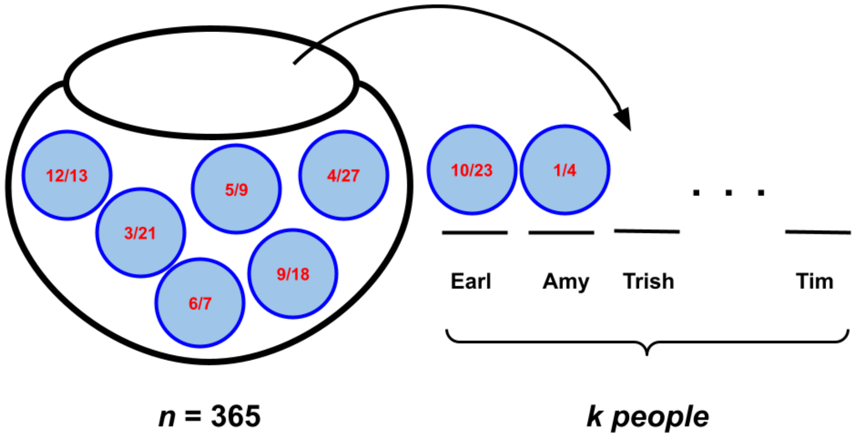

Figure 2.7

Example 2.10. **(The birthday problem)** Imagine that k random people convene for a birthday party, and consider two related questions:

- What is the probability that *at least* 2 of the attendees share a birthday?
- How large does the party need to be for the above probability to be at least 50%?

We begin by making a few assumptions:

- There are 365 days in the year (so we ignore leap day);
- A random attendee is equally likely to have any of the 365 days as their birthday (this is empirically false, but oh well);
- The party is not being held at a popular restaurant or a laser tag place or any other such place where you would expect many birthday guys and gals to descend at the same time;
- The attendees are *independent*. If two folks have the same birthday, it is pure coincidence.

We will ultimately be invoking $P(A) = \#(A)/\#(S)$, and our event A is "*at least* 2 of the attendees share a birthday," so what is the sample space S? It is the set of all possible assignments of birthdays to k people. Figure 2.7 visualizes what we have in mind. We have k people (slots) at this party, and their birthdays have been randomly assigned to them as if we were drawing them from the proverbial urn; each ball is one of the possible days of the year. So to compute $\#(S)$, we are selecting k from $n = 365$. But how?

- This is **with replacement** because two people could have been born on the same day. Indeed, if we ruled this possibility out, there would be nothing for us to study. So when a birthday comes out, it goes back in the jar to potentially be drawn again;
- **Order matters**. In Figure 2.7 we have labeled the slots with the names of the unique attendees. This makes each slot distinct, and the order in which the balls (dates) are drawn determines which person receives which date. We consider Trish on 9/3 and Tim on 4/30 a different outcome from Trish on 4/30 and Tim on 9/3, and so order matters here.

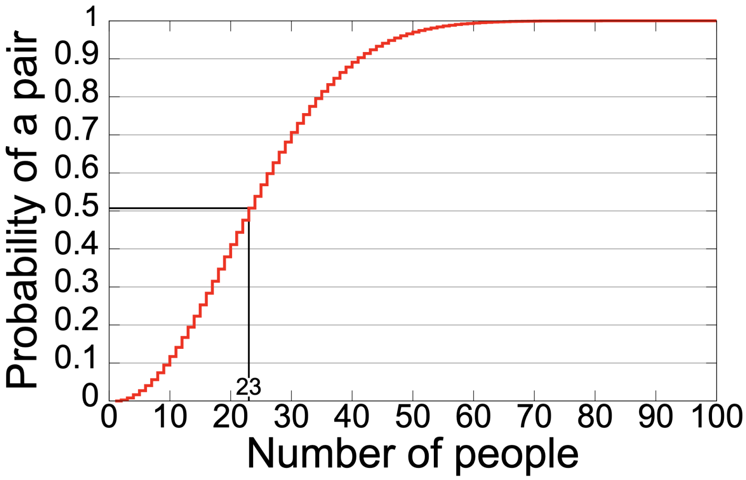

Figure 2.8

k	$P(A)$	$\binom{k}{2}$
1	0.0	0
2	0.0027	1
5	0.027	10
10	0.116	45
15	0.2529	105
23	0.5072	253
32	0.753	496
56	0.988	1540
60	0.994	1770
100	0.9999	4950
$\vdots$	$\vdots$	$\vdots$
366	1.0	66795

Table 2.5

With that, we know from Table 2.3 that $\#(S) = 365^k$. So what is $\#(A)$? Start by registering that "*at least* 2" is spooky language. There are several (too many) ways that *at least two* people could share a birthday. Let's use the complement rule to simplify:

$$P(A) = 1 - P(A^c) = 1 - \frac{\#(A^c)}{\#(S)} = 1 - \frac{\#(A^c)}{365^k}.$$

A^c is "no one shares a birthday," which is much simpler to count. How many ways can we assign birthdays to k people such that no one shares? Well, order continues to matter, but now we are drawing *without* replacement. Once a birthday is assigned, it will not be assigned again, thus ensuring that there are no duplicates. As such, $\#(A^c) = 365!/(365-k)!$, and so

$$P(A) = 1 - P(A^c) = 1 - \frac{365!}{365^k(365-k)!}.$$

Table 2.5 displays this probability for various values of k, the number of partygoers. When $k = 1$, then $P(A) = 0$ because that one person has no one they could match with. By the time $k = 366$ attend, at least one match is guaranteed as a consequence of the *pigeonhole principle*. Imagine 365 people are in attendance, and by some miracle there are no birthday matches; every person has a unique birthday. As soon as a 366th person arrives at the party, their birthday *must* match with someone. All of the possible days of the year have already been exhausted.

So those are the extreme cases of $k = 1$ and $k = 366$. But newcomers tend to be surprised by what happens when $1 < k < 366$. It only takes $k = 23$ for a 50% chance of at least one match, and by the time $k = 60$, it's all but guaranteed. Folks often guess that you need a large number of attendees (on the order of hundreds) for the probability of a match to be high, but what they fail to realize is that every unique pair of people is a fresh opportunity for a match. Table 2.5 shows the number of unique pairs for each k, and while k may be small, the number of pairs gets large pretty quickly, making at least one match more and more likely.

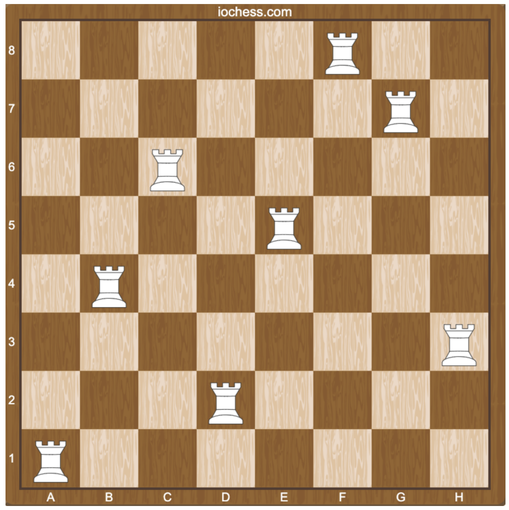

Figure 2.9: Eight safe rooks.

Example 2.11. (**Eight rooks**) Imagine we take eight identical rooks and randomly place them on *distinct* squares on an 8×8 chess board. What is the probability that all of the rooks are "safe" from one another? Figure 2.9 visualizes an example of this. Recall that rooks can move up and down or side-to-side, but not diagonally. In order to be safe from one another, two rooks therefore need to occupy different rows *and* columns of the board. For the eight rooks to be mutually safe, they all must be in different columns and different rows from one another. This is a job for counting. There are a finite number of ways to place the rooks on the board, and all placements are equally likely, so we must count the total number of placements and the total number of *safe* placements. There are $n = 8 \cdot 8 = 64$ squares on the board, and we are selecting $k = 8$ of them at which to place the rooks. The rooks are identical, so order doesn't matter, and we have $\#(S) = \binom{64}{8}$. The event we care about is $A =$ "the rooks are safe from one another," so how many ways can this happen? We know that a safe placement will have every row and column occupied by exactly one rook, so let us imagine building up a safe placement row-wise, starting from the bottom and working upward. There are 8 ways to place a single rook on the bottom row. No other rooks can go there, so we move to the next row. There are only 7 ways to place the next rook in the next row, because we must avoid the column occupied by the first rook. Similarly, there are only 6 ways to place the next rook in the next row, because we must avoid the two columns occupied by the first two rooks. And so on. As a consequence of the counting principle, where we regard each row as an experiment, we see that $\#(A) = 8 \times 7 \times 6 \times \cdots \times 2 \times 1 = 8!$. So $P(A) = 8!/\binom{64}{8} \approx 0.0000091095$.