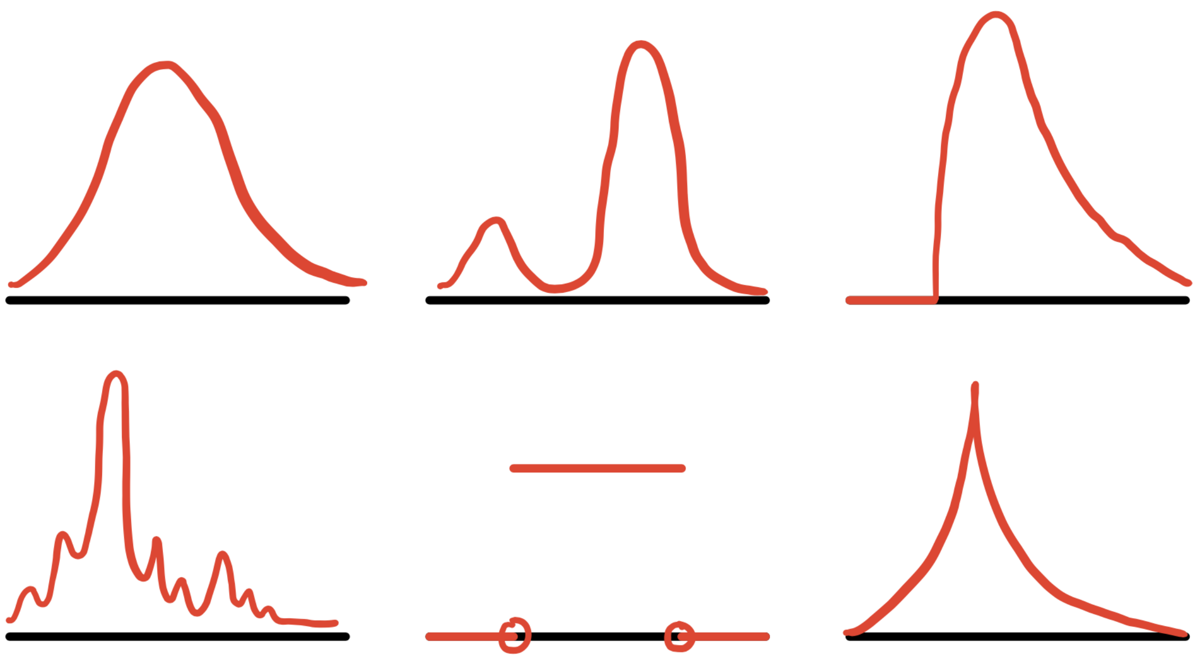

Figure 3.16: Cartoons of what a probability density function might look like. Area under the curve corresponds to probabilities. The total area underneath the pdf must be one.

3.6 Absolutely continuous random variables

Definition 3.13. A random variable X is **absolutely continuous** if its cdf $F_X(x) = P(X \leq x)$ is an absolutely continuous function for all $x \in \mathbb{R}$.

The concept of absolute continuity is beyond the scope of our course. You can look it up if you wish, but all I want you to know is that absolute continuity is a special kind of continuity that is stronger, and it ensures that the cdf is sufficiently smooth as to rule out the pathological cases

By ruling out such cases and requiring the cdf to be extra smooth, we guarantee the existence of a special object, which is the continuous analog of the pmf

Theorem 3.10. If X is an absolutely continuous random variable, then there exists a function $f_X : \mathbb{R} \to \mathbb{R}$ called the **probability density function** (**pdf**) with the following properties:

- $f_X(x) \geq 0$ for all $x \in \mathbb{R}$;
- $F'_X(x) = f_X(x)$;
- $F_X(x) = \int_{-\infty}^{x} f_X(t)\,\mathrm{d}t$;
- $P(a < X < b) = \int_a^b f_X(x)\,\mathrm{d}x = F_X(b) - F_X(a)$ (fundamental theorem of calculus!);
- $\int_{-\infty}^{\infty} f_X(x)\,\mathrm{d}x = P(-\infty < X < \infty) = 1$;

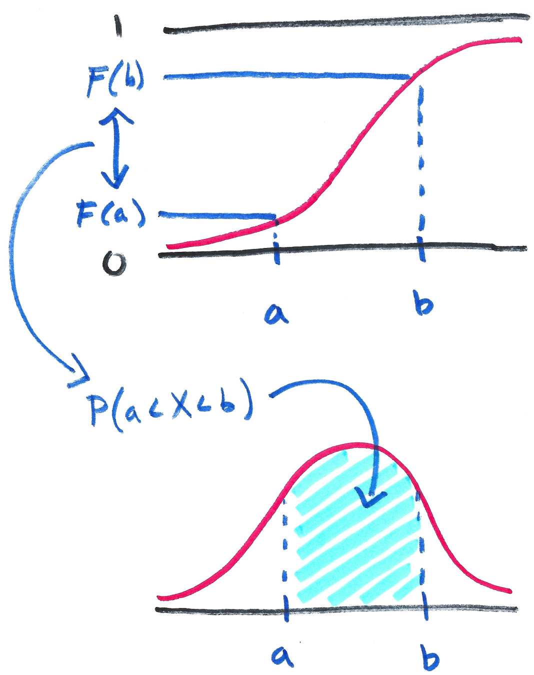

Figure 3.17: $P(a < X < b) = \int_a^b f_X(x)\,\mathrm{d}x = F_X(b) - F_X(a)$

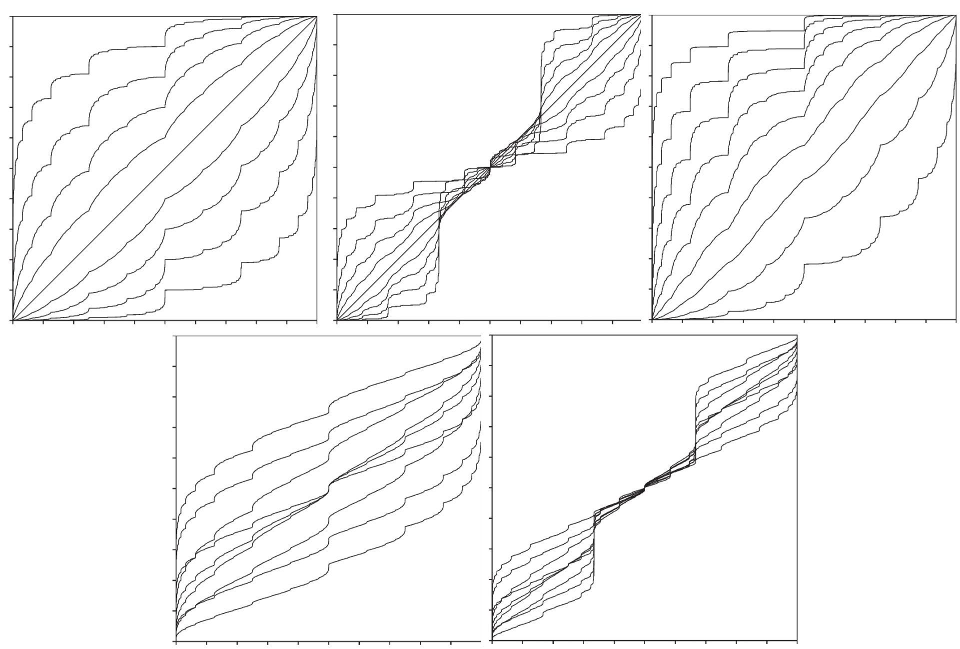

Figure 3.18: Examples of singular distributions.

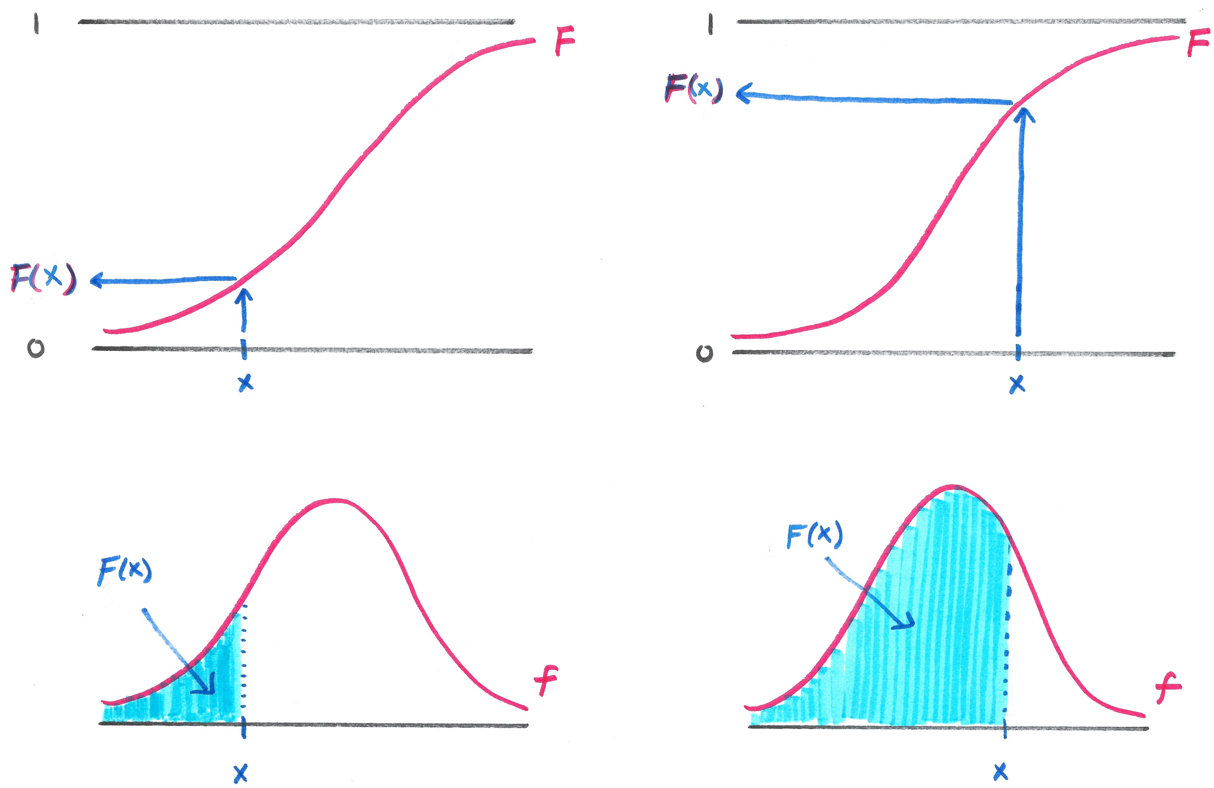

Figure 3.19: the cdf F is tracking the amount of area (probability) under the pdf f that we accumulate as we move from left to right. When we start on the left with "$x = -\infty$," we've accumulated none of it, and as $x \to \infty$, we eventually accumulate all of it (total measure 1).

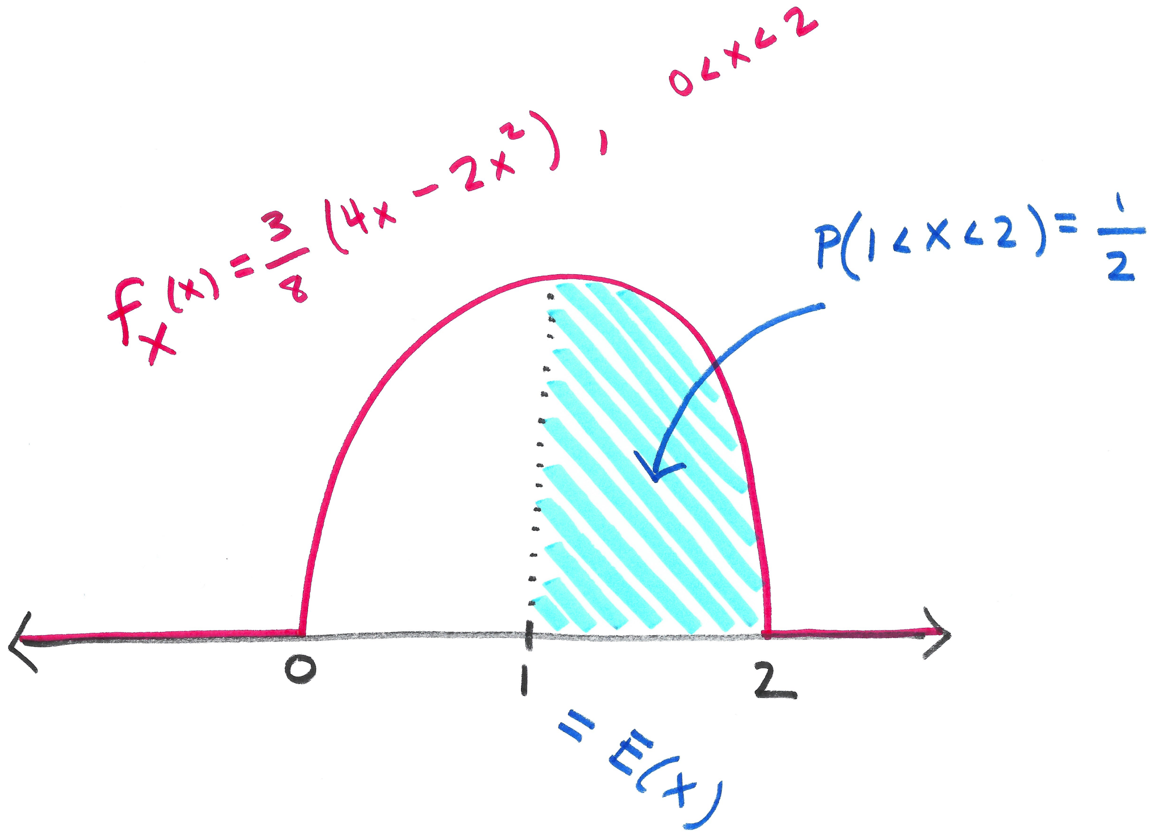

Figure 3.20

Figure 3.16 displays cartoons of what this function might look like, and Figure 3.17 illustrates how probabilities are calculated.

Remark 3.9. Absolutely continuous random variables are the continuous analog to discrete random variables, and the pdf plays the role of the pmf. In discrete world, we *summed* the pmf to compute probabilities. In continuous world, we *integrate* the pdf. In discrete world, we *differenced* the cdf to compute the pmf. In continuous world, we *differentiate* it.

Remark 3.10. The relationship $F_X(x) = \int_{-\infty}^{x} f_X(t)\,\mathrm{d}t$ means that the cdf is the **area accumulation function** of the pdf. This is illustrated in Figure 3.19.

Remark 3.11. I hasten to emphasize: not all continuous random variables have a pdf. Random variables whose cdf is continuous but not absolutely continuous are called **singular**, and Figure 3.18 displays some examples of what that might look like. The Cantor distribution in Figure 3.15 is the most famous example of this. The derivative of the Cantor distribution is zero (almost) everywhere, so a density with the above properties could not possibly exist. When the pdf does exist however, it's super useful, because it allows us to compute probabilities and moments by *integrating*.

Example 3.14. Let X be absolutely continuous with density

$$f_X(x) = \begin{cases} c(4x - 2x^2) & 0 < x < 2 \\ 0 & \text{else.} \end{cases}$$

A cartoon is displayed in Figure 3.20. The range of X is the set of real numbers where the density is nonzero. This is called the **support** of the density function:

$$\text{supp}(f_X) = \{x \in \mathbb{R} : f_X(x) > 0\}, \tag{3.15}$$

and so $\text{Range}(X) = \text{supp}(f_X)$ for absolutely continuous random variables, and in this case we see that $\text{Range}(X) = (0, 2)$. And recall that because X is continuous, we need not quibble about whether or not we include the endpoints. The constant c is a **normalizing constant** that serves to ensure that the density integrates to 1, so let's find its value. First, note that

$$\begin{aligned} \int_{-\infty}^{\infty} f_X(x)\,\mathrm{d}x &= \int_{-\infty}^{0} f_X(x)\,\mathrm{d}x + \int_{0}^{2} f_X(x)\,\mathrm{d}x + \int_{2}^{\infty} f_X(x)\,\mathrm{d}x \\ &= \cancel{\int_{-\infty}^{0} 0\,\mathrm{d}x} + \int_{0}^{2} c(4x - 2x^2)\,\mathrm{d}x + \cancel{\int_{2}^{\infty} 0\,\mathrm{d}x} \\ &= c\int_{0}^{2} (4x - 2x^2)\,\mathrm{d}x \end{aligned}$$

So

$$\begin{aligned} \int_{-\infty}^{\infty} f_X(x)\,\mathrm{d}x = 1 &\implies c\int_{0}^{2} (4x - 2x^2)\,\mathrm{d}x = 1 \\ &\implies c = \frac{1}{\int_0^2 (4x - 2x^2)\,\mathrm{d}x}. \end{aligned}$$

As such:

$$\int_0^2 (4x - 2x^2)\,\mathrm{d}x = \left[2x^2 - \frac{2}{3}x^3\right]_0^2 = 2 \cdot 4 - \frac{2}{3} \cdot 8 = 8/3.$$

So $c = 1/(8/3) = 3/8$. If we wanted to compute $P(X > 1)$, we can do that by integrating the pdf:

$$\begin{aligned} P(X > 1) &= P(X \in (1, \infty)) \\ &= \int_1^{\infty} f_X(x)\,\mathrm{d}x \\ &= \int_1^{2} f_X(x)\,\mathrm{d}x + \int_2^{\infty} f_X(x)\,\mathrm{d}x \\ &= \int_1^2 \frac{3}{8}(4x - 2x^2)\,\mathrm{d}x + \cancel{\int_2^{\infty} 0\,\mathrm{d}x} \\ &= \frac{3}{8}\left[2x^2 - \frac{2}{3}x^3\right]_1^2 \\ &= \frac{3}{8}\left[2 \cdot 4 - \frac{2}{3} \cdot 8 - 2 + \frac{2}{3}\right] \\ &= \frac{1}{2}. \end{aligned}$$

Definition 3.14. If X is absolutely continuous with density f, then the expected value of X is

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,\mathrm{d}x. \tag{3.16}$$

As in the discrete case, the expected value may not exist or be finite.

Example 3.15. Continuing Example 3.14, We see that

$$\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x f_X(x)\,\mathrm{d}x \\
&= \int_{-\infty}^{0} x f_X(x)\,\mathrm{d}x + \int_{0}^{2} x f_X(x)\,\mathrm{d}x + \int_{2}^{\infty} x f_X(x)\,\mathrm{d}x \\
&= \cancel{\int_{-\infty}^{0} x \cdot 0\,\mathrm{d}x} + \int_{0}^{2} x \frac{3}{8}(4x - 2x^2)\,\mathrm{d}x + \cancel{\int_{2}^{\infty} x \cdot 0\,\mathrm{d}x} \\
&= \frac{3}{8}\int_{0}^{2} (4x^2 - 2x^3)\,\mathrm{d}x \\
&= \frac{3}{8}\left[\frac{4}{3}x^3 - \frac{2}{4}x^4\right]_0^2 \\
&= \frac{3}{8}\left[\frac{4}{3}8 - \frac{2}{4}16\right] \\
&= 1.
\end{aligned}$$

It makes sense that this is the mean because the pdf is symmetric about 1.

Theorem 3.11. (**LOTUS, again**) If X is absolutely continuous with density f_X and $g : \text{Range}(X) \to \mathbb{R}$ is a transformation, then

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,\mathrm{d}x. \tag{3.17}$$

The upshot of this result is the same as before: in order to compute $E[g(X)]$, it is not necessary to first derive the entire density of the transformed variable $g(X)$. We can make due with the original density f_X. All of the results in Section 3.3 about the linearity of expectation and the variance apply unmodified to continuous and absolutely continuous random variables. They apply to all random variables regardless of type, in fact:

- $E(aX + b) = aE(X) + b$;
- $E\left(\sum_{i=1}^{n} c_i X_i\right) = \sum_{i=1}^{n} c_i E(X_i)$, no matter how the X_i are related;
- $\text{var}(X) = E[(X - E(X))^2] = E(X^2) - E(X)^2$;
- $\text{var}(aX + b) = a^2\text{var}(X)$.
- If the X_i are independent, then $\text{var}\left(\sum_{i=1}^{n} c_i X_i\right) = \sum_{i=1}^{n} c_i^2 \text{var}(X_i)$.

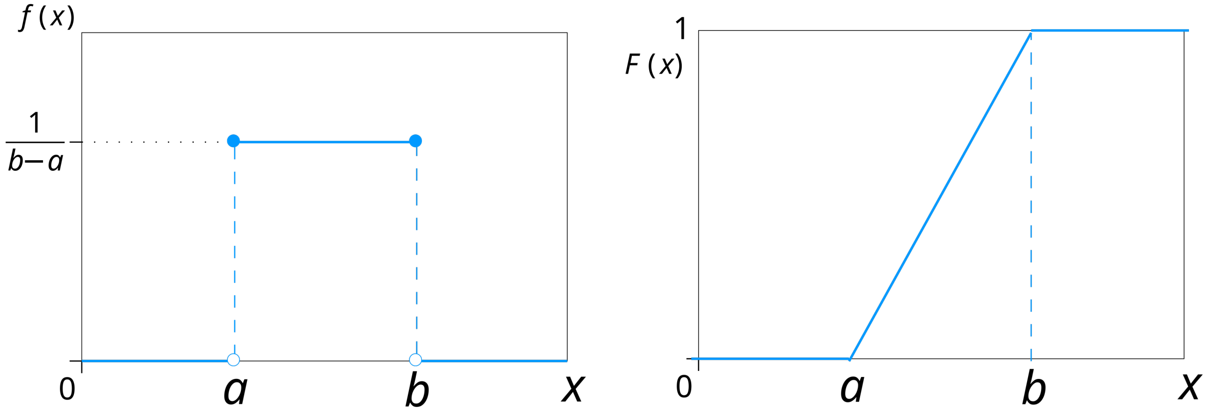

Figure 3.21: pdf and cdf of Unif(a, b).

3.6.1 Uniform distribution

Definition 3.15. X has the **continuous uniform distribution** on the interval Range(X) $=$ $[a, b]$ if the density is

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{else.} \end{cases} \tag{3.18}$$

This density is pictured in Figure 3.21, and we denote this $X \sim \text{Unif}(a, b)$.

Compare this with the discrete uniform distribution from Section 3.2.5. In that case, "uniform" meant every every value in the range had the same probability. That is trivially true for all continuous random variables because they have $P(X = x) = 0$ for all $x \in \mathbb{R}$, so "uniform" has a slightly different meaning in the continuous case. It means that all tiny lil' sub-intervals of the same length have the same probability. So if $(c, d), (u, v) \subseteq (a, b)$ have the same length $d - c = v - u$, then $P(c < X < d) = P(u < X < v) = (d - c)/(b - a) = (v - u)/(b - a)$. This is illustrated in Figure 3.22

Does the pdf integrate to 1?

It had better. Let's check:

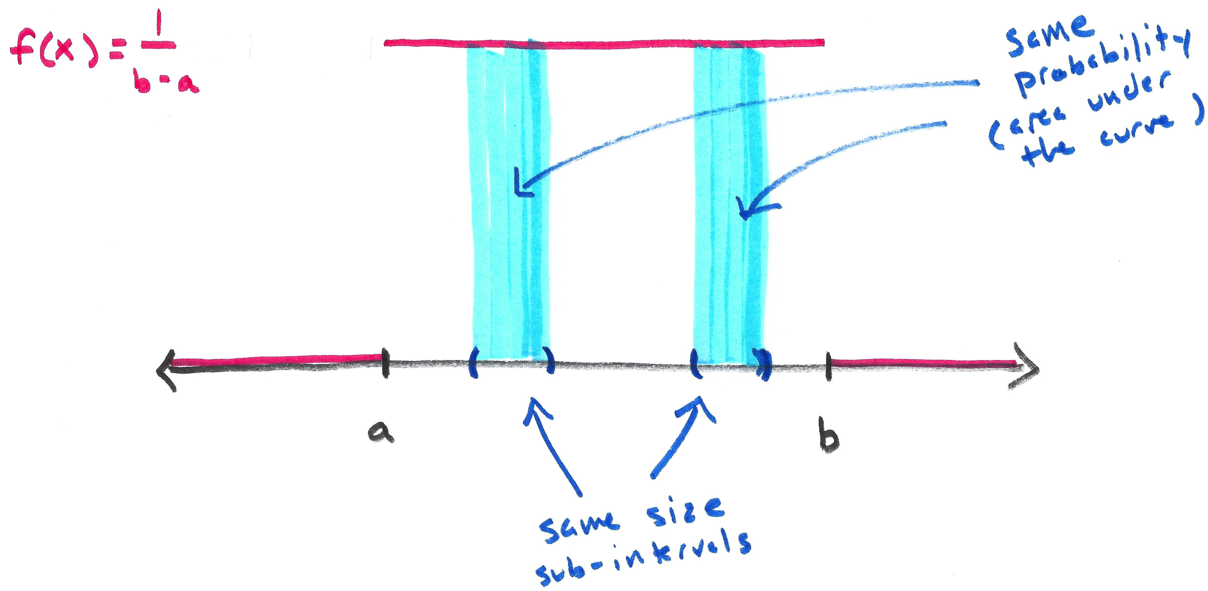

Figure 3.22: The continuous uniform distribution is "uniform" in the sense that sub-intervals of the same length have the same probability, regardless where they are located.

$$
\begin{aligned}
\int_{-\infty}^{\infty} f_X(x)\,\mathrm{d}x &= \int_{-\infty}^{a} f_X(x)\,\mathrm{d}x + \int_{a}^{b} f_X(x)\,\mathrm{d}x + \int_{b}^{\infty} f_X(x)\,\mathrm{d}x \\
&= \cancel{\int_{-\infty}^{a} 0\,\mathrm{d}x} + \int_{a}^{b} \frac{1}{b-a}\,\mathrm{d}x + \cancel{\int_{b}^{\infty} 0\,\mathrm{d}x} \\
&= \frac{1}{b-a}\int_{a}^{b} 1\,\mathrm{d}x \\
&= \frac{1}{b-a}[x]_a^b \\
&= \frac{1}{b-a}(b-a) \\
&= 1.
\end{aligned}
$$

What is the cdf?

The cdf is $F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(t)\,\mathrm{d}t$. If $x < a$, then

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,\mathrm{d}t = \int_{-\infty}^{x} 0\,\mathrm{d}t = 0.$$

If $a \leq x \leq b$, then

$$\begin{aligned}
F_X(x) &= \int_{-\infty}^{x} f_X(t)\,\mathrm{d}t \\
&= \int_{-\infty}^{a} f_X(t)\,\mathrm{d}t + \int_{a}^{x} f_X(t)\,\mathrm{d}t \\
&= \cancel{\int_{-\infty}^{a} 0\,\mathrm{d}t} + \int_{a}^{x} \frac{1}{b-a}\,\mathrm{d}t \\
&= \frac{1}{b-a}[t]_a^x \\
&= \frac{x-a}{b-a}.
\end{aligned}$$

If $b < x$, then

$$\begin{aligned}
F_X(x) &= \int_{-\infty}^{x} f_X(t)\,\mathrm{d}t \\
&= \int_{-\infty}^{a} f_X(t)\,\mathrm{d}t + \int_{a}^{b} f_X(t)\,\mathrm{d}t + \int_{b}^{x} f_X(t)\,\mathrm{d}t \\
&= \cancel{\int_{-\infty}^{a} 0\,\mathrm{d}t} + \int_{a}^{b} \frac{1}{b-a}\,\mathrm{d}t + \cancel{\int_{b}^{x} 0\,\mathrm{d}t} \\
&= 1.
\end{aligned}$$

So the cdf of $X \sim \text{Unif}(a,\, b)$ is

$$F_X(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & b < x. \end{cases} \tag{3.19}$$

This is displayed in Figure 3.21.

What is the mean?

$$
\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x f_X(x)\,\mathrm{d}x \\
&= \int_{-\infty}^{a} x f_X(x)\,\mathrm{d}x + \int_{a}^{b} x f_X(x)\,\mathrm{d}x + \int_{b}^{\infty} x f_X(x)\,\mathrm{d}x \\
&= \cancel{\int_{-\infty}^{a} x \cdot 0\,\mathrm{d}x} + \int_{a}^{b} x \frac{1}{b-a}\,\mathrm{d}x + \cancel{\int_{b}^{\infty} x \cdot 0\,\mathrm{d}x} \\
&= \frac{1}{b-a} \int_{a}^{b} x\,\mathrm{d}x \\
&= \frac{1}{b-a} \left[\frac{1}{2}x^2\right]_a^b \\
&= \frac{1}{b-a} \left[\frac{1}{2}b^2 - \frac{1}{2}a^2\right] \\
&= \frac{1}{2(b-a)} \left[b^2 - a^2\right] \\
&= \frac{1}{2\cancel{(b-a)}}\cancel{(b-a)}(b+a) \\
&= \frac{b+a}{2}.
\end{aligned}
$$

So the expected value is the midpoint of the interval $[a,\ b]$. This makes sense because the uniform distribution on an interval is symmetric around its midpoint.